Semester – IV
CC-8 (Mathematical Physics – III)

Complex Analysis

1. (a) Show that $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ for all $z_1, z_2 \in \mathbb{C}$.

   (b) verify that $\sqrt{2}\,|z| \geqslant |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$

   (c) Sketch the curves in the complex plane is given by (i) $\mathrm{Im}(z) = -1$ (ii) $|z-1| = |z+i|$ (iii) $2|z| = |z-2|$

2. (a) Express the following in the form $x+iy$ with $x, y \in R$

   (i) $\dfrac{i}{1-i} + \dfrac{1-i}{i}$ (ii) all the 3rd roots of $-8i$

   (iii) $\left(\dfrac{i+1}{\sqrt{2}}\right)^{1337}$.

   (b) Find the principle argument and exponential form of (i) $z = \dfrac{i}{1+i}$ (ii) $z = \sqrt{3} + i$ (iii) $z = 2 - i$

   (c) Find all the complex roots of the equations:

   (i) $z^6 = -9$ (ii) $z^2 + 2z + (1-i) = 0$ (iii) $z^4 + 16 = 0$.

3. (a) Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x+iy$. Use the expressions $x = \dfrac{z+\bar{z}}{2}$ and $y = \dfrac{z-\bar{z}}{2i}$ to write $f(z)$ in terms of z and simplify the result.

   (b) Show that (i) $|\sin z|^2 = (\sin x)^2 + (\sinh y)^2$

   $|\cos z|^2 = (\cos x)^2 + (\sinh y)^2$

   (c) Find all the complex roots of the equation $\cos z = 3$.

   (d) calculate (i) $\sin(\pi/4 + i)$ (ii) $\cos(\pi/3 + i)$.

4. (a) Find i^i and its principle value.

   (b) Let $f(z)$ be the principle branch of $\sqrt[3]{z}$

   (i) Find $f(-i)$ (ii) Show that $f(z_1)f(z_2) = \lambda f(z_1 z_2)$ for all $z_1, z_2 \neq 0$ where $\lambda = 1, \dfrac{-1+\sqrt{3}i}{2}$ or $\dfrac{-1-\sqrt{3}i}{2}$.

   (c) Let $f(z)$ be the principle branch of z^{-i}

   (i) Find $f(i)$ (ii) show that $f(z_1)f(z_2) = \lambda f(z_1 z_2)$ for all $z_1, z_2 \neq 0$ where $\lambda = 1, e^{2\pi}$ or $e^{-2\pi}$.

5. (a) Test the analyticity of the functions given below
(i) $\omega = \ln z$ (b) $\omega = \frac{1}{z}$ (c) $\omega = \bar{z}$

(b) Show that both the real and the imaginary parts of the complex functions (i) $\omega = z^3$ and (ii) $\sin z$ satisfy CR and Laplace conditions.

(c) An analytic function $f(z)$ has its real part $e^{-x}(x\cos y + y\sin y)$ and $f(0) = 1$. Show that $f(z) = 1 + ze^{-z}$.

(d) Verify that the following functions u are harmonic and in each case give a conjugate harmonic function v (ie v such that $u+iv$ is analytic)
(i) $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$
(ii) $u(x,y) = \ln(x^2+y^2)$.

6. (a) Compute the following contour integral
$\int_L \bar{z}\,dz$ where L is the boundary of the triangle ABC with $A=0$, $B=1$ and $C=i$; oriented counter clockwise.

(b) Evaluate the contour integral
$\int_C f(z)\,dz$ using the parametric representations for C where $f(z) = \frac{z^2-1}{z}$ and the curve C is
(i) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$
(ii) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$
(iii) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

7. (a) Let C be the boundary of the triangle with vertices at the points 0, $3i$ and -4 oriented counterclockwise. Compute the contour integral
$\int_C (e^z - \bar{z})\,dz$

(b) Apply Cauchy Integral theorem to show that

$$\int_C f(z)\,dz = 0 \quad \text{when } C \text{ is the unit circle } |z|=1$$

in either direction and when

(i) $f(z) = \dfrac{z^3}{z^2+5z+6}$

(ii) $f(z) = \tan z$

(iii) $f(z) = \text{Log}(z+3i)$

(c) Let C_1 denote the positively oriented boundary of the curve given by $|x|+|y| = 2$ and C_2 be the positively oriented circle $|z| = 4$. Apply Cauchy Integral Theorem to show that $\int_{C} f(z)\,dz = \int_{C_2} f(z)\,dz$ when

(i) $f(z) = \dfrac{z+1}{z^2+1}$ (ii) $f(z) = \dfrac{z+2}{\sin(z/2)}$ (iii) $f(z) = \dfrac{\sin z}{z^2+6z+5}$

(d) Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals using Cauchy integral formula

(i) $\int_C \dfrac{z\,dz}{z+1}$ (ii) $\int_C \dfrac{\cosh z}{z^2+z}\,dz$ (iii) $\int_C \dfrac{\tan(z/2)}{z-\pi/2}\,dz$.

8. (a) Find the value of the integral $f(z)$ around the circle $|z-i| = 2$ oriented counterclockwise when

(i) $f(z) = \dfrac{1}{z^2+4}$ (ii) $f(z) = \dfrac{1}{z(z^2+4)}$

(b) Let C be the circle $|z|=1$ oriented counterclockwise

(i) Compute $\int_C \dfrac{1}{z^2-8z+1}\,dz$

(ii) Using part (i) compute $\int_0^{\pi} \dfrac{1}{4-\cos\theta}\,d\theta$

(c) Compute the integral $\int_{-\pi}^{\pi} \dfrac{dx}{2-(\cos x+\sin x)}$.

(d) Compute the integral $\int_0^{\infty} \dfrac{x\,dx}{x^3+1}$.

9. (a) Find the Taylor series of the following function and their radii of convergence

(i) $z \sinh(z^2)$ at $z=0$.

(ii) e^z at $z=2$

(iii) $\dfrac{z^2+z}{(1-z)^2}$ at $z=-1$

(b) Find the Taylor series of $(\cos z)^2$ at $z=\pi$.

(c) Find a power series expansion of the function $f(z)=\frac{1}{3-z}$ about the point $4i$ and calculate the radious of convergence.

10. (a) Find a Laurent-series expansion of the function $f(z) = z^{-1}\sinh(z^{-1})$ about the point 0, and classify the singularity at 0.

(b) Consider the function $f(z) = \dfrac{\sin z}{\cos(z^3)-1}$.
classify the singularity at $z=0$ and calculate the residue.

(c) Let $f(z) = \dfrac{z^2}{z^2-z-2}$. Find the Laurent series of $f(z)$ in each of the following domain

(i) $1<|z|<2$ (ii) $0<|z-2|<1$

11. (a) For each of the following complex functions, do the following:

- find all its singularities in $\mathbb{C}$
- Write the principle part of the function at each singularity
- for each singularity, determine whether it is pole, a removable singularities or essential singularity.
- Compute the residue of the function at each singularity

(i) $f(z) = \dfrac{1}{(\cos z)^2}$ (ii) $f(z) = (1-z^3)\exp\left(\frac{1}{z}\right)$

(iii) $f(z) = \dfrac{e^z}{1-z^2}$ (iv) $f(z) = \dfrac{\sin z}{z^{2010}}$

(v) $f(z) = (1-z^2)\exp\left(\frac{1}{z}\right)$ (vi) $f(z) = \dfrac{1}{(\sin z)^2}$

(vii) $f(z) = \dfrac{1-\cos z}{z^2}$ (viii) $f(z) = \dfrac{e^z}{z(z-1)^2}$ (ix) $f(z) = \dfrac{\cos z}{z^2 - z^3}$

Application of Residue theorem

(12) (a) Calculate $\int_{C} \frac{8-z}{z(4-z)} dz$ where C is the circle of radious 7, centre 0, negatively oriented.

(Application of Cauchy Residue theorem)

(b) compute the integral $\int_{0}^{\pi} \frac{d\theta}{2-\cos\theta}$.

(c) Let $a, b \in \mathbb{R}$ such that $a^2 > b^2$. Calculate the integral

$$\int_{0}^{\pi} \frac{d\theta}{a+b\cos\theta}$$

(d) Using the method of residues Evaluate

$$\int_{0}^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}.$$

13. (a) Determine the poles of following function and residue at each pole

$$f(z) = \frac{4-3z}{z(z-1)^2(z-2)^2}$$ and hence evaluate

$$\oint_{C} \frac{4-3z}{z(z-1)(z-2)}.$$

(b) Evaluate $\int_{0}^{\infty} \frac{\cos mx}{x^2+1} dx$.

(c) Show that $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \pi/3$.

(d) Show that $\int_{-\infty}^{\infty} \frac{x \sin \pi x \, dx}{x^2+2x+5} = -\pi e^{-2\pi}$.

Integral Transformations

1) (a) Find the Fourier transform of the function

$$f(x) = \begin{cases} 1, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$

(b) Find the Fourier sine and cosine transforms of $f(x) = e^{-ax}$.

(c) Find the Fourier transform of the function $f(x) = e^{-x^2/2}$.

2) (a) Prove the following:

(i) $F_s[x f(x)] = -\frac{d}{dw} F_c(w)$ (ii) $F_c[w f(x)] = \frac{d}{dw} F_s(w)$

(b) using the above relations Find the Fourier cosine and sine transform of xe^{-ax}.

(c) Find the Fourier transform of the function

$$f(x) = \begin{cases} 1-x^2, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

3) (a) Find the Fourier sine transform of the following function

$$f(x) = \begin{cases} x^2, & \text{when } 0 \le x < \pi \\ 0 & \text{elsewhere.} \end{cases}$$

(b) Find the Fourier cosine transform of

$$f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

(c) prove that if a given function is shifted in either direction by an amount a, no Fourier component changes in amplitude, the change is confined to a phase change only.

4) (a) state and prove the parseval's relations or identities of Fourier transform.

(b) Apply parseval's identity to evaluate the integral

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

(c) Given that if

$$f(x) = \begin{cases} 1 & \text{for } |x| < 0 \\ 0 & \text{for } |x| > a > 0 \end{cases}$$

then $F(s) = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$

Taking this result and using parseval's identity, show that

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$$

5. (a) If the Fourier transform of $f(x)$ be $\frac{1-\cos n\pi}{n^2\pi^2}$ $(0 < x < \pi)$, find $f(x)$

(b) Find the Fourier transform of $f(x)$ ~~be~~ $f(x) = e^{-a|x|}$. Hence show that

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}, \quad m > 0.$$

(c) Solve the following integral equation

$$\int_0^\infty f(x) \alpha x \, dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ \alpha, & \alpha > 1 \end{cases}$$

Hence evaluate the integral $\int_0^\infty \frac{\sin^2 t}{t^2} dt$.

6. (a) using parseval's identity, show that

$$\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

(b) Apply Parseval's identity to evaluate the integral

$$\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx.$$

(c) Solve for $f(x)$ from the integral equation

$$\int_0^\infty f(x) \cos wx \, dx = e^{-w}.$$

7. (a) Consider $f(x) \to 0$ for $x \to \pm\alpha$. Let Fourier transform of $f(x)$ be $g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(s)\, e^{iks}\, ds$.

If $g_1(k)$ is the Fourier transform of $\frac{df}{dx}$ then show that $g_1(k) = -ik\, g(k)$

(b) If $\phi(s)$ is the Fourier sine transform of $f(x)$ for $s > 0$ then show that $F_s\{f(x)\} = -\phi(-s)$ for $s < 0$.

(c) Find the Fourier sine transformation of $\frac{e^{-\lambda x}}{x}$.

8. (a) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0, t > 0$ subject to the conditions (use the method of FT)

(i) $u = 0$ when $x = 0, t > 0$

(ii) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$ when $t = 0$

(iii) $u(x,t)$ is bounded, $\frac{\partial u}{\partial x} \to 0$ as $x \to \alpha$.

(b) Use the method of FT to determine the displacement $y(x,t)$ of an infinite string, given that the string is initially at rest and that the initial displacement is $f(x)$, $-\alpha < x < \alpha$. Show that the solution can also be put in the form $y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$.

(c) The current $I(t)$ in a simple circuit containing the resistance R and inductance L satisfies the equation $L\frac{dI}{dt} + RI = E(t)$

where $E(t)$ is the applied electromagnetic force given by $E(t) = E_0 \exp(-a|t|)$. Find $I(t)$.
(use the method of FT).

Introduction to Probability & Spl. Theory of Relativity

Short questions (2 marks each):

1. In water, can a electron travel faster than light ?
2. What will be the velocity of a particle if its kinetic energy is equal to relativistic kinetic energy ?
3. Does the density of an object change as its speed increases? If yes, by what factor ?
4. Calculate the rest mass, relativistic mass and momentum of a photon having energy 5 eV.
5. A muon at rest has lifetime 2.0 x 10^{-6}s. What is its life time when it travels with a velocity 3c/5 ?
6. How fast must a 2 m stick be moving if its length is observed to be 1 m from laboratory frame ?
7. Derive a relation between relativistic momentum p, relativistic kinetic energy k and rest mass m_0. Starting from the equation $E=mc^2$.
8. Suppose a circle of radius b is set in motion. Calculate the relativistic speed parameter v/c such that the circle is seen as an ellipse of semi-minor axis a and semi-major axis b, where a<b.
9. Starting from Einstein's velocity addition theorem show that the expression leads to corresponding Newtonian/Galilean expression in the limit velocity (v) goes to infinity.
10. What is proper time ? Explain why it is used to define four velocity.
11. Staring from the definition of four velocity, construct the four momentum and explain the significance of 0^{th} and space components of it in Relativistic and non-relativistic limit.
12. Prove that STR forbids photoelectric effect for a free electron.
13. What do you mean by elastic, inelastic and explosive collisions in STR ?
14. Using properties of dot product in Minkowski space for products of four momentum prove that $E^2=p^2c^2+m_0^2c^4$.
15. What do you mean by random experiment and random variable ?
16. Find the probability of getting exact two and at least two heads in tossing of an unbiased coin thrice.
17. Find the value of n and p given the mean and variance of the Binomial distribution are 4 and 8/3.
18. A fair die is thrown 600 times. Let W be the number of times '2' occurs. Find the expectation of 'W'.
19. Show that for a random variable following normal distribution, the probability of acquiring a value greater than the mean is ½.
20. Can a probability density function f(x) can acquire a value greater than 1 for some x ? Explain.

Long Questions (Theoretical and Numerical) each questions carry 10 marks:

1. State the postulates of STR. Using the postulates and some thought experiments obtain the time dilation length contraction.
2. Using the expressions for length contraction and time dilation obtain the Lorentz transformation equation.
3. Starting from L.T., express the L.T. in terms of hyperbolic rotations involving x and t and determine the rotation angle in terms of v/c.
4. In a frame S the following two events occur. Event 1 : x1=x0, t1=x_0/c and y1=z1=0. Event 2 : x2=2x0, t2=x_0/2c$ and y2=z2=0. Find the velocity of the frame S' (w.r.t. S) at which these two events occur simultaneously. What is the value of t' in S' at which these two events are simultaneous? x0 is a constant and c is speed of light in free space. Also determine the velocity of the frame where they occur at the same point. Discuss the physical significance of the result. Hence define time-like, space-like and light-like interval.

5. A space traveler with speed v synchronizes his clock (t'=0) with his earth friend (t=0). The earth man then observes both clocks simultaneously, t directly and t' through a telescope. What does t read when t' reads one hour ? A light beam is propagating through a block of glass with index of refraction n. If the block is moving at constant velocity v in the same direction as the beam, what is the speed of light in the block as measured by an observer in the laboratory? In the non-relativistic limit the results matches with our perception of Newtonian mechanics.
6. (a) Show that Lorentz transformation can be regarded as a rotation of axes (t-x) through an imaginary angle given $\theta=\tan^{-1}(i\,\beta)$, where $\beta=v/c$. (b) Show that the ordering of events will remain same in two inertial frames moving with uniform speed relative to each other provided that it is not possible to send any signal with speed greater than the speed of light provided they are separated by time-like interval. (c) Two lumps of clay each of rest mass m0 move towards each other with equal speed 4c/5 and stick together. What is the mass of the lump ?
7. (a) Starting from the expression for force as the rate of change of relativistic momentum and using the expression $E=T+m_0c^2$ for energy (in usual notation), show that the acceleration is not always parallel to the force. Hence obtain expressions for longitudinal and transverse mass. (b)Two rods each of proper length L0 move lengthwise towards each other parallel to the common axis with the same velocity v relative to the laboratory frame. Find the expression for the length of the other rod as observed from the frame each rod.
8. What is twin-paradox ? How is it resolved ? What is light cone ? Explain the physical significance of it. Sketch a light-cone a geometrically discuss the significance of events being separated by time-like and space-like intervals.
9. Two protons collide with each other to create three protons and one anti-proton. Discuss the above process in the following two scenario and obtain the minimum energy required in each of the cases (a) One proton is has total energy E and the other is at rest in lab frame.($E_T=E+m_0c^2$) (b) Two protons collide each other in lab frame and each of them has total energy E $E_T=E+E=2E$. Discuss why lesser value of energy (E_T) required in the second case.
10. Starting from LT derive the Einstein’s theorem for velocity addition. Using it prove that that maximum velocity attainable by a particle is c. Prove that creation of electron-positron pair is not possible from a single photon in absolute vacuum.
11. Discuss a thought experiment to derive the expression for relativistic Doppler effect. Using the above expression and a suitable thought experiment deduce $E=mc^2$.
12. Discuss the Michelson Morley experiment. Show that how the null result rejected the ether hypothesis. Show that how STR is consistent with the findings of the M-M experiment.
13. Show that two simultaneous events at different positions in a frame of reference are not in general simultaneous in another frame connected by L.T. What happens if the frames are connected by Galilean transformation ? A meter stick is held at 45 degree with the direction of motion in a system moving with a velocity 0.8c. What is the length measured in the laboratory frame ? (c) Show that the four dimensional volume element dxdydzdt is invariant under L.T. A body of mass m at rest breaks up simultaneously into two parts with masses m1 and m2 and speeds v1 and v2 respectively. Show that m > m1 + m2, using conservation of mass-energy.
14. Starting from $E=mc^2$ derive an expression for relativistic kinetic energy and show that in the non relativistic limit it reduces to Newtonian expression. Define the position and velocity four vectors (both covariant and contravariant forms). Hence discuss the metric tensor in Minkowski space time. What is the physical significance of metric tensor ?
15. Using the p.m.f for Binomial distribution derive the expectation and variance of a binomial distribution (n,p). Show that variance is less than mean. Determine relative fluctuation defined as s.d./mean.

16. Using the p.m.f for Poisson distribution derive the expectation and variance of a poisson distribution (λ). Prove that a binomial (n,p) tends to a Poisson (λ=np), in the limit p tends to 0 and n tends to infinity with np being a finite constant.
17. If X follows a binomial (n,p) prove that $P(X=even)$ is $=0.5\{1+(q-p)^n\}$, where $p+q=1$. In a population 85% of the people have Rh positive blood group. Suppose that two people get married. What is the probability that both of them are Rh negative ?
18. Suppose that the probability of a customer buying an egg roll is 0.6. If there are 5 customers in a line and 2 egg rolls are already prepared, what is the probability that a customer will have to wait for an egg-roll ? What is the probability that exactly 1 customer will have to wait for an egg-roll ?
19. (a)The average number of traffic accidents on a section of high way is two per week. Assume that the number of accidents follows a Poisson distribution with mean 2. Find the probability of at most three accidents during two week period. (b)At a petrol pump automobiles arrivals are assumed to follow Poisson with average rate of 50 per hour. If the petrol pump has only one pump and all car requires one minute to obtain fuel, find the probability that car will have to wait for getting the fuel.
20. (a)If a random variable X follows normal distribution with mean 10 and s.d. 50. Find $P(X\leq-108)$, $P(|X|\geq108)$,$P(10<X\leq108)$,$P(X\leq108|X>10)$. Given $\Phi(1.96)=0.9750021$ and $\Phi(2.36)=0.9908625$. (b)Show that a normal distribution is symmetric about its mean and determine the mode of a normal distribution.